A Mixed Quadrature Rule by blending Lobatto rule and Modified Clenshaw-Curtis Rule due to Richardson Extrapolation

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Abstract

A mixed quadrature rule of precision nine for approximate evaluation of real definite integrals has been constructed by blending Clenshaw-Curtis five point quadrature rule modified by Richardson Extrapolation quadrature method and Lobatto five point quadrature rule. The mixed quadrature rule has been tested and is found to more effective than that of constituent lower precision rules. The mixed quadrature rule can be use in adaptive environment.

Key words: *Clenshaw-Curtis quadrature rule, Lobatto five point rule, Richardson Extrapolation, Mixed quadrature rule,* $SM_9(f), L_5(f), CC_5(f), RCC_5(f)$.

1. Introduction

Real definite integral of the type

$$I(f) = \int_a^b f(x)dx \tag{1.1}$$

have been successfully approximated by several Authors [1], [2], [3], [4] by applying the mixed quadrature rule. Recently, real definite integral of type (1.1) has been successfully approximated by some authors by applying the mixed quadrature rule. The method involves construction of a quadrature rule of higher precision by linear/convex combinations of two other rules of equal lower precision. In literature, precision of a quadrature rule has been enhanced through Richardson extrapolation and Kronrod extension. These methods of precision enhancement are very much cumbersome and each having single base rule. But the enhancement of precision by mixed quadrature approach is very much simple with the aid of two rules and easy to handle.

In this paper, keeping in view the improvement of precision method proposed by earlier authors, a mixed quadrature rule of precision nine has been designed by the linear combination of following two methods of different precision and using the technique of Richardson extrapolation on (I).

I. Clenshaw-curtis 5-point rule ($CC_5(f)$ is of precision-5)
II. Lobatto 5-point rule ($L_5(f)$ $is\ of\ precision - 7$)

I. The Clenshaw-Curtis Quadrature Rule

The Clenshaw-Curtis method essentially approximate a function $f(t)$ over any interval $[a-h,\ a+h]$ using Chebyshev polynomials $T_r(x)$ of degree n

$$f(t) = F(x) = \sum_{r=0}^{n}{}' a_r T_r(x) \qquad (-1 \le x \le 1) \tag{1.2}$$

where a_r are the expansion coefficient and $\sum{}'$ denotes a finite sum whose 1st term is to be halved before beginning to the sum. That is

$$F(x) = \frac{1}{2}a_0T_0(x) + a_1T_1(x) + a_2T_2(x) + \cdots a_nT_n(x) \tag{1.3}$$

Collecting with $f(a+hx)$ at (n+1) points

$$x_i = cos\left(\frac{i\pi}{n}\right), \qquad i = 0,1,\ldots n$$

one can evaluate the expansion coefficients a_r.

The Chebyshev polynomials $T_r(x)$ can be expressed as

$$T_r(x_i) = cos(r\cos^{-1}(x_i)) = \cos\left(\frac{ri\pi}{n}\right) \qquad r \ge 0$$

Then $\sum_{i=0}^{n}{}'' f(a+hx_i)T_r(x_i) = \sum_{i=0}^{n}{}'' \sum_{k=0}^{n}{}' a_kT_k(x_i)\, T_r(x_i) = \sum_{k=0}^{n}{}' a_k \sum_{k=0}^{n}{}'' \cos\left(\frac{ki\pi}{n}\right)\cos\left(\frac{ri\pi}{n}\right)$ (1.4)

$\sum{}''$ means that the 1st and last terms are to be halved before the summation begins.

The orthogonality of cosine function with respect to the points $x_i = cos\left(\frac{i\pi}{n}\right)$ is expressed by

$$\sum_{k=0}^{n}{}'' \cos\left(\frac{ki\pi}{n}\right)\cos\left(\frac{ri\pi}{n}\right) = \begin{cases} n & r = k = 0 \text{ or } n \\ \frac{n}{2} & 0 < r = k < n \\ 0 & r \ne k \end{cases} \tag{1.5}$$

Substituting values from (1.5) into (1.4), we obtain

$$\sum_{i=0}^{n}{}'' f(a+hx_i)T_r(x_i) = \begin{cases} na_r & r = k = n \\ \frac{n}{2}a_r & 0 \le r = k < n \\ 0 & r \ne k \end{cases}$$

Hence

$$a_r = \begin{cases} \frac{2}{n}\sum_{i=0}^{n}{}'' f(a+hx_i)T_r(x_i) & r = 0,1,2\ldots n-1 \\ \frac{1}{n}\sum_{i=0}^{n}{}'' f(a+hx_i)T_r(x_i) & r = n \end{cases} \tag{1.6}$$

Write

$$I(f) = \int_{a-h}^{a+h} f(t)dt \tag{1.7}$$

On substituting $t = a + hx$, we get $I(f) = \int_{-1}^{1} f(x)dx$

Assuming $I = I_n$, we obtain

$$I_n = h\int_{-1}^{1}\sum_{r=0}^{n\ '} a_r T_r(x)dx = h\sum_{r=0}^{n\ '} a_r\int_{-1}^{1} T_r(x)dx$$

Substituting the values of (as obtain (1.6)), we get

$$I_n = h\sum_{r=0}^{n\ ''}\frac{2}{n}\sum_{i=0}^{n\ ''} f(a+hx_i)T_r(x_i)\int_{-1}^{1} T_r(x)dx = h\sum_{r=0}^{n\ ''}\frac{2}{n}\sum_{i=0}^{n\ ''} f(a+hx_i)T_r(x_i)\frac{-2}{r^2-1}$$

[Since $\int_{-1}^{1} T_r(x)dx = \frac{-2}{r^2-1}$, r is even]

We can write

$$I_n = h\sum_{r=0}^{n\ ''} w_i f(a+hx_i) \qquad (1.8)$$

where $w_i = -\frac{4}{n}\sum_{r=0}^{n\ ''} T_r(x_i)\frac{1}{r^2-1}$ i=0,1,2…n

(1.8) is known as Clenshaw-Curtis (n+1) quadrature rule.

For n=4, we can obtain the Clenshaw-Curtis 5-point rule ($CC_5(f)$)

$$I_4 = CC_5(f) = \frac{h}{15}\left[f(a+h) + 8f\left(a+\frac{h}{\sqrt{2}}\right) + 12f(a) + 8f\left(a-\frac{h}{\sqrt{2}}\right) + f(a-h)\right] \qquad (1.9)$$

2. Clenshaw-Curtis 5-point rule

Consider a real valued function $f(x)$ and $[a-h, a+h] \subseteq dom(f)$

Let $I(f) = \int_{a-h}^{a+h} f(x)dx$ (2.1)

Using Clenshaw-Curtis 5-point rule ($CC_5(f)$) for evaluation of the integral (2.1)

we have

$$I(f) \approx CC_5(f) = \frac{h}{15}\left[f(a+h) + 8f\left(a+\frac{h}{\sqrt{2}}\right) + 12f(a) + 8f\left(a-\frac{h}{\sqrt{2}}\right) + f(a-h)\right] \qquad (2.2)$$

Applying Taylor's Theorem, after simplification we obtain

$$CC_5(f) = 2h\left[f(a) + \frac{h^2}{3!}f^{ii}(a) + \frac{h^4}{5!}f^{iv}(a) + \frac{2}{15}\frac{h^6}{6!}f^{vi}(a) + \frac{1}{10}\frac{h^8}{8!}f^{viii}(a) + \frac{1}{12}\frac{h^{10}}{10!}f^{x}(a) + \frac{3}{40}\frac{h^{12}}{12!}f^{xii}(a) + \cdots\right] \qquad (2.3)$$

The exact value of the integral

$$I(f) = 2h\left[f(a) + \frac{h^2}{3!}f^{ii}(a) + \frac{h^4}{5!}f^{iv}(a) + \frac{h^6}{7!}f^{vi}(a) + \frac{h^8}{9!}f^{viii}(a) + \frac{h^{10}}{11!}f^{x}(a) + \frac{h^{12}}{13!}f^{xii}(a) + \cdots\right] \qquad (2.4)$$

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Error of the $CC_5(f)$ rule

Let us denote the error due to Clenshaw-Curtis 5-point rule for approximating the integral (2.1) by $ECC_5(f)$

Thus $\quad I(f) = CC_5(f) + ECC_5(f)$

$$\Rightarrow ECC_5(f) = I(f) + CC_5(f) \tag{2.5}$$

Using the (2.3) and (2.4) on (2.5), after simplification we obtain

$$ECC_5(f) = \frac{2}{15}\frac{h^7}{7!}f^{vi}(a) + \frac{1}{5}\frac{h^8}{9!}f^{viii}(a) + \frac{1}{6}\frac{h^{11}}{11!}f^{x}(a) + \frac{1}{20}\frac{h^{13}}{13!}f^{xii}(a) + \cdots \tag{2.6}$$

From the error expression (2.6) it is clear that the degree of precision of the Clenshaw-Curtis 5-point rule is five.

3. Modified Clenshaw-Curtis 5 point rule due to Richardson Extrapolation $RCC_5(f)$

We have

$$CC_5(f) = \frac{h}{15}\left[f(a+h) + 8f\left(a+\frac{h}{\sqrt{2}}\right) + 12f(a) + 8f\left(a-\frac{h}{\sqrt{2}}\right) + f(a-h)\right]$$

Changing the step length we have

$$CC_{5\frac{h}{2}}(f) = \frac{2h}{15}\left[f(a+2h) + 8f\left(a+\sqrt{2}h\right) + 12f(a) + 8f\left(a-\sqrt{2}h\right) + f(a-2h)\right] \tag{3.1}$$

Applying Taylor's Theorem

$$CC_{5\frac{h}{2}}(f) = 4h\left[f(a) + \frac{(2h)^2}{3!}f^{ii}(a) + \frac{(2h)^4}{5!}f^{iv}(a) + \frac{128}{15}\frac{h^6}{6!}f^{vi}(a) + \frac{384}{15}\frac{h^8}{8!}f^{viii}(a) + \frac{1280}{15}\frac{h^{10}}{10!}f^{x}(a) + \frac{4608}{15}\frac{h^{12}}{12!}f^{xii}(a) + \cdots\right] \tag{3.2}$$

Error associated due to change of the step length $ECC_{5\frac{h}{2}}(f)$ is given by

$$ECC_{5\frac{h}{2}}(f) = I(f) - CC_{5\frac{h}{2}}(f)$$

Using (3.1) and (2.3)

$$ECC_{5\frac{h}{2}}(f) = \frac{256}{15}\frac{h^7}{7!}f^{vi}(a) + \frac{1536}{15}\frac{h^8}{9!}f^{viii}(a) + \frac{5120}{15}\frac{h^{11}}{11!}f^{x}(a) + \frac{6144}{15}\frac{h^{13}}{13!}f^{xii}(a) + \cdots \tag{3.3}$$

Now
$$I(f) = CC_5(f) + ECC_5(f) \tag{3.4}$$

and
$$I(f) = CC_{5\frac{h}{2}}(f) + ECC_{5\frac{h}{2}}(f) \tag{3.5}$$

Multiplying 128 with (3.4) and subtracting (3.5) from the result, we have

$$(128-1)I(f) = \left[128CC_5(f) - CC_{5\frac{h}{2}}(f)\right] + \left[128ECC_5(f) - ECC_{5\frac{h}{2}}(f)\right]$$

$$\Rightarrow I(f) = \frac{1}{127}\left[128CC_5(f) - CC_{5\frac{h}{2}}(f)\right] + \frac{1}{127}\left[128ECC_5(f) - ECC_{5\frac{h}{2}}(f)\right]$$

$$= RCC_5(f) + ERCC_5(f)$$

Where

$$RCC_5(f) = \frac{1}{127}\left[128CC_5(f) - CC_{5\frac{h}{2}}(f)\right] \quad (3.6)$$

and $$ERCC_5(f) = \frac{1}{127}\left[128ECC_5(f) - ECC_{5\frac{h}{2}}(f)\right]$$

Now using (2.6) and (3.3)

$$ERCC_5(f) = \frac{1}{127}\left[128\left[\frac{2}{15}\frac{h^7}{7!}f^{vi}(a) + \frac{1}{5}\frac{h^8}{9!}f^{viii}(a) + \frac{1}{6}\frac{h^{11}}{11!}f^{x}(a) + \frac{1}{20}\frac{h^{13}}{13!}f^{xii}(a) + \cdots\right]\right.$$
$$\left. - \left[\frac{256}{15}\frac{h^7}{7!}f^{vi}(a) + \frac{1536}{15}\frac{h^8}{9!}f^{viii}(a) + \frac{5120}{15}\frac{h^{11}}{11!}f^{x}(a) + \frac{6144}{15}\frac{h^{13}}{13!}f^{xii}(a) + \cdots\right]\right]$$

$$= -\frac{384}{127\times 5}\frac{h^8}{9!}f^{viii}(a) - \frac{320}{127}\frac{h^{11}}{11!}f^{x}(a) - \frac{2016}{127\times 5}\frac{h^{13}}{13!}f^{xii}(a) + \cdots \quad (3.7)$$

(3.6) and (3.7) are respectively called Modified Clenshaw-Curtis rule and Error in Modified Clenshaw-Curtis rule due to Richardson extrapolation.

From (3.7), the degree of precision of the rule is 7.

4. (II) Lobatto 5-point rule

The Lobatto 5-point rule is given by

$$L_5(f) = \frac{h}{90}\left[9\{f(a-h) + f(a+h)\} + 49\left\{f\left(a - \frac{\sqrt{3}}{\sqrt{7}}h\right) + f\left(a + \frac{\sqrt{3}}{\sqrt{7}}h\right)\right\} + 64f(a)\right] \quad (4.1)$$

applying Taylors theorem, we have

$$L_5(f) = \frac{2h}{90}\left[90f(a) + 30\frac{h^2}{2!}f^{ii}(a) + 18\frac{h^4}{4!}f^{iv}(a) + \frac{90}{7}\frac{h^6}{6!}f^{vi}(a) + \frac{522}{49}\frac{h^8}{8!}f^{viii}(a)\right.$$
$$\left. + \frac{3330}{343}\frac{h^{10}}{10!}f^{x}(a) + \frac{22338}{2401}\frac{h^{12}}{12!}f^{xii}(a) + \cdots\right]$$

$$\Rightarrow L_5(f) = 2h\left[f(a) + \frac{h^2}{3!}f^{ii}(a) + \frac{h^4}{5!}f^{iv}(a) + \frac{h^6}{7!}f^{vi}(a) + \frac{29}{5\times 49}\frac{h^8}{8!}f^{viii}(a) + \frac{37}{343}\frac{h^{10}}{10!}f^{x}(a) + \right.$$
$$\left. \frac{1241}{5\times 2401}\frac{h^{12}}{12!}f^{xii}(a) + \cdots\right] \quad (4.2)$$

Denoting the truncation error of Lobatto 5 point rule by $\boldsymbol{EL_5(f)}$,

$$I(f) = L_5(f) + EL_5(f)$$

$$\Rightarrow EL_5(f) = I(f) - L_5(f) \quad (4.3)$$

Using (2.4) and (4.2) on (4.3) we obtain

$$EL_5(f) = 2h\left[f(a) + \frac{h^2}{3!}f^{ii}(a) + \frac{h^4}{5!}f^{iv}(a) + \frac{h^6}{7!}f^{vi}(a) + \frac{h^8}{9!}f^{viii}(a) + \frac{h^{10}}{11!}f^{x}(a) + \frac{h^{12}}{13!}f^{xii}(a) + \cdots\right] - 2h\left[f(a) + \frac{h^2}{3!}f^{ii}(a) + \frac{h^4}{5!}f^{iv}(a) + \frac{h^6}{7!}f^{vi}(a) + \frac{29}{5\times 49}\frac{h^8}{8!}f^{viii}(a) + \frac{37}{343}\frac{h^{10}}{10!}f^{x}(a) + \frac{1241}{5\times 2401}\frac{h^{12}}{12!}f^{xii}(a) + \cdots\right]$$

$$\Rightarrow EL_5(f) = -\frac{32}{245}\frac{h^9}{9!}f^{viii}(a) - \frac{128}{343}\frac{h^{11}}{11!}f^{x}(a) - \frac{8256}{12005}\frac{h^{13}}{13!}f^{xii}(a) - \cdots \quad (4.4)$$

The error term established that the degree of precision of $L_5(f)$ is 7.

5. Construction of the mixed quadrature rule of precision nine

For the construction of proposed mixed quadrature rule let us consider $L_5(f)$ and $RCC_5(f)$.

We have $$I(f) = RCC_5(f) + ERCC_5(f) \quad (5.1)$$

and $$I(f) = L_5(f) + EL_5(f) \quad (5.2)$$

Now multiplying (5.1) and (5.2) by 127 and 588 respectively, then subtracting the 1st result from the 2nd result, we obtain

$$461I(f) = [588L_5(f) - 127RCC_5(f)] + [588EL_5(f) - 127ERCC_5(f)]$$

$$\Rightarrow I(f) = \frac{1}{461}[588L_5(f) - 127RCC_5(f)] + \frac{1}{461}[588EL_5(f) - 127ERCC_5(f)]$$

$$\Longrightarrow I(f) = SM_9(f) + ESM_9(f)$$

Where $$SM_9(f) = \frac{1}{461}[588L_5(f) - 127RCC_5(f)]$$

Or $$SM_9(f) = \frac{1}{461}\left[588L_5(f) - 128CC_5(f) + CC_{5\frac{h}{2}}(f)\right] \quad \text{[using (3.6)]} \quad (5.3)$$

Using (2.2), (3.1) and (4.1) on (5.3), we obtained

$$SM_9(f) = \frac{h}{6945}\left[2\{f(a-2h) + f(a+2h)\} - 155\{f(a-h) + f(a+h)\} + 16\{f(a-\sqrt{2}h) + f(a+\sqrt{2}h)\} - 1024\left\{f\left(a-\frac{h}{\sqrt{2}}\right) + f(a+\frac{h}{\sqrt{2}})\right\}\right] + \frac{h}{20745}\left[11221\left\{f\left(a-\frac{\sqrt{3}}{\sqrt{7}}h\right) + f(a+\frac{\sqrt{3}}{\sqrt{7}}h)\right\} - 1020f(a)\right] \quad (5.4)$$

$SM_9(f)$ is the desired mixed quadrature rule, (5.3) be its general form and (5.4) be its Extension form. The degree of precision of the rule is 9 which established by the theorem-1. The truncation error generated by $SM_9(f)$ is given by

$$ESM_9(f) = \frac{1}{461}[588EL_5(f) - 127ERCC_5(f)] \quad (5.5)$$

Theorem-1

If $f(x)$ is sufficiently differentiable in the interval $[a-h, a+h]$, the degree of precision of the rule $SM_9(f)$ is 9 and $ESM_9(f) = o(h^{11})$.

Proof

Consider the truncation error generated by $SM_9(f)$, given by (5.5)

$$ESM_9(f) = \frac{1}{461}[588EL_5(f) - 127ERCC_5(f)]$$

Now using (3.7) and (4.4) in the error term, we obtain

$$ESM_9(f) = -\frac{704}{3227}\frac{h^{11}}{11!}f^{x}(a) - \frac{288}{245}\frac{h^{13}}{13!}f^{xii}(a) + \cdots \quad (5.6)$$

This established that the degree of precision of the rule $SM_9(f)$ is 9 and $ESM_9(f) = o(h^{11})$. □

Theorem-2 (*Error Analysis*)

The error committed due to the mixed quadrature rule $SM_9(f)$ is less than from which it formulated

Proof:

From (2.6) and (5.6) $|ECC_5(f)| \leq |ESM_9(f)|$

From (3.7) and (5.6) $|ERCC_5(f)| \leq |ESM_9(f)|$

From (4.4) and (5.6) $|EL_5(f)| \leq |ESM_9(f)|$ □

Numerical verification

The effectiveness of the rule is verified by applying it in different integrals given in the table-6.1

Table-6.1

Sl no	*Integrals*	*Exact value*	*Value by quadrature rules*			*Error approximated*		
			$CC_5(f)$	$L_5(f)$	$SM_9(f)$	$\|ECC_5(f)\|$	$\|EL_5(f)\|$	$\|ESM_9(f)\|$
1	$\int_0^1 \frac{1}{1+e^x}dx$	0.379885493041	0.3798854926	0.379885493049	0.379885493037	4.41×10^{-10}	8×10^{-12}	4×10^{-12}
2	$\int_1^2 \frac{1}{1+x^4}dx$	0.2031547018	0.2031548432	0.2031548460	0.2031547876	1.597×10^{-7}	1.25×10^{-7}	8.58×10^{-8}
3	$\int_0^{\pi/2} \frac{sinx\, dx}{(1+cosx)^3}$	0.375	0.3749998904	0.3750000760	0.3750000443	1.096×10^{-7}	7.6×10^{-8}	4.43×10^{-8}
4	$\int_0^2 \frac{x}{1+x^3}dx$	0.7237976339	0.7237975821	0.7237977086	0.7237976801	5.18×10^{-8}	7.47×10^{-8}	4.62×10^{-8}
5	$\int_0^{\pi/2} \frac{dx}{1+cosx}$	1	0.9999998240	1.0000000671	1.0000000588	1.76×10^{-7}	6.71×10^{-8}	5.88×10^{-8}

6. Conclusions

From the table it is evident that the mixed quadrature rule when applied each of the five integrals gives better result than that of constituent rules (Lobatto five point rule and Clenshaw-Curtis 5 point rule). The quadrature rule $SM_9(f)$ *reduces the number of steps required to approximate an integral can be cheek by using adaptive environment. This mixed quadrature rule also use for computing Adaptive quadrature.*

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